Continuous Sparse Recovery for Direction of Arrival Estimation with Co-prime Arrays

Zhao Tan, Arye Nehorai
Department of Electrical and Systems Engineering
Washington University in St. Louis
St. Louis, MO 63130
Email: {tanz,nehorai}@ese.wustl.edu

Yonina C. Eldar
Department of Electrical Engineering
Technion—Israel Institute of Technology
Haifa 32000, Isreal
E-mail: yonina@ee.technion.ac.il

***Abstract*—We consider the problem of direction of arrival (DOA) estimation using a newly proposed structure of co-prime arrays. A continuous sparse recovery method is implemented in order to increase resolution. We show that in the noiseless case one can theoretically detect up to $\frac{MN}{2}$ sources with only $2M+N$ sensors via continuous sparse recovery. The noise statistics of co-prime arrays are also analyzed to demonstrate the robustness of the proposed optimization scheme. Using numerical examples, we show the superiority of the proposed method.**

***Index Terms*—Direction of arrival estimation, co-prime arrays, continuous sparse recovery method**

I. Introduction

Co-prime arrays were proposed in [1] to increase the degrees of freedom of the array by studying the covariance matrix of the received signals among different sensors. It was shown that by using $O(M+N)$ sensors, this structure can achieve $O(MN)$ degrees of freedom. In [2] the authors proved that an increased number of sources can be detected by co-prime arrays using MUSIC with spatial smoothing. However, the application of spatial smoothing reduces the obtained virtual array aperture [3]. Another methodology uses sparsity based recovery to overcome this disadvantage of subspace methods [3]. The assumption made by sparsity methods is that all sources are located exactly on the predefined grid points. Off-grid targets can deteriorate the performance of sparse recovery significantly [4]. In [5], the joint sparsity between the original signal and grid mismatches was exploited during the DOA estimation for co-prime arrays. Due to the first order approximation used in [5], the estimation performance is still limited by the higher order modeling error.

To overcome the difficulty of convential sparse recovery, a recently developed continuous sparse recovery method [6], [7] is utilized in this paper to perform DOA estimation with co-prime arrays. One merit of this method is that it considers all the possible locations within the interested range, and thus does not suffer from modeling error. Using this framework we demonstrate theoretically that with $2M+N$ sensors, co-prime arrays can detect up to $\frac{MN}{2}$ sources. We then establish the robustness of the proposed method under Gaussian noise.

Throughout the paper, we use capital italic bold letters to represent matrices and operators, and lowercase italic bold letters to represent vectors. For a given matrix $\boldsymbol{A}$, $\boldsymbol{A}^*$ denotes the conjugate transpose matrix, $\boldsymbol{A}^{\mathrm{T}}$ denotes the transpose, and $\boldsymbol{A}^{\mathrm{H}}$ represents the conjugate matrix without transpose. We use A_{mn} to denote the (m,n)th element of $\boldsymbol{A}$. We use $\otimes$ to denote the Kronecker product of two matrices. For a given operator $\boldsymbol{F}$, $\boldsymbol{F}^*$ denotes the conjugate operator of $\boldsymbol{F}$. Given a vector $\boldsymbol{x}$, we use $\|\boldsymbol{x}\|_1$ and $\|\boldsymbol{x}\|_2$ to denote its ℓ_1 and ℓ_2 norms; x_i and $x[i]$ are both used to represent the ith element of $\boldsymbol{x}$. For a function f, $\|f\|_{L_1}, \|f\|_{L_2}, \|f\|_{L_\infty}$ are its $\ell_1, \ell_2, \ell_\infty$ norms.

II. DOA Estimation With Co-Prime Arrays

Consider a linear array with L sensors which may be non-uniformly located. Assume that there are K narrow band sources located at $\theta_1, \theta_2, \ldots, \theta_K$ with signal powers $\sigma_1^2, \sigma_2^2, \ldots, \sigma_K^2$. The steering vector for the kth source located at θ_k is $\boldsymbol{a}(\theta_k) \in \mathbb{C}^{L\times 1}$ with lth element $e^{\boldsymbol{j}(2\pi/\lambda)d_l \sin(\theta_k)}$, in which d_l is the location of the lth sensor and λ is the wavelength. The data collected by all sensors at time t can be expressed as

$$\boldsymbol{x}(t) = \sum_{k=1}^{K} \boldsymbol{a}(\theta_k) y_k(t) + \boldsymbol{\varepsilon}(t), \tag{1}$$

for $t = 1, \ldots, T$, in which $\boldsymbol{\varepsilon}(t) \in \mathbb{C}^{L\times 1}$ is an i.i.d. white Gaussian noise $\mathcal{CN}(0, \sigma^2)$, $\boldsymbol{x}$ is a vector where each element is the data at the lth sensor, and $y_k(t)$ represents the transmitted signal from the kth source at time t, which is distributed as $\mathcal{CN}(0, \sigma_k^2)$. We assume that the sources are temporally uncorrelated.

The correlation matrix among the L sensors can then be expressed as

$$\boldsymbol{R}_{xx} = E[\boldsymbol{x}(t)\boldsymbol{x}^*(t)] = \sum_{k=1}^{K} \sigma_k^2 \boldsymbol{a}(\theta_k)\boldsymbol{a}^*(\theta_k) + \sigma^2 \boldsymbol{I}. \tag{2}$$

After vectorizing the covariance matrix $\boldsymbol{R}_{xx}$, we have

$$\boldsymbol{z} = \mathrm{vec}(\boldsymbol{R_{xx}}) = \boldsymbol{\Phi}(\theta_1, \theta_2, \ldots, \theta_K)\boldsymbol{s} + \sigma^2 \mathbf{1}_n, \tag{3}$$

The work of Z. Tan and A. Nehorai was supported by the AFOSR Grant FA9550-11-1-0210, and ONR Grant N000141310050. The work of Y. C. Eldar was supported in part by the Israel Science Foundation under Grant no. 170/10, in part by the Ollendorf Foundation, and in part by a Magneton from the Israel Ministry of Industry and Trade.

where $\boldsymbol{\Phi}(\theta_1,\ldots,\theta_K) = [\boldsymbol{a}(\theta_1)^{\mathrm{H}} \otimes \boldsymbol{a}(\theta_1),\ldots,\boldsymbol{a}(\theta_K)^{\mathrm{H}} \otimes \boldsymbol{a}(\theta_K)]$. $\boldsymbol{s} = [\sigma_1^2,\sigma_2^2,\ldots,\sigma_K^2]$, and $\mathbf{1}_n = [\boldsymbol{e}_1^{\mathrm{T}},\boldsymbol{e}_2^{\mathrm{T}},\ldots,\boldsymbol{e}_L^{\mathrm{T}}]^{\mathrm{T}}$, where $\boldsymbol{e}_i$ denotes a vector with all zero elements, except for the ith element, which equals to one.

From equation (3), we see that $\boldsymbol{s}$ behaves like a coherent source and $\sigma^2\mathbf{1}_n$ becomes a deterministic noise term. The distinct rows in $\boldsymbol{\Phi}$ act as a larger virtual array with sensors located at $d_i - d_j$, with $1 \le i,j \le L$. Consider a co-prime array structure consisting of two arrays with N and $2M$ sensors respectively. The locations of the N sensors are in the set $\{Mnd, 0 \le n \le N-1\}$, and the locations of the $2M$ sensors are in the set $\{Nmd, 0 \le m \le 2M-1\}$.The locations of the virtual sensors in (3) are then given by the cross difference set $\{\pm(Mn - Nm)d, 0 \le n \le N-1, 0 \le m \le 2M-1\}$ and the two self difference sets. In order to implement continuous sparse recovery, we are interested in generating a consecutive range of virtual sensors. It was shown in [2] that when M and N are coprime numbers, a consecutive range can be created from $-MNd$ to MNd from the virtual array.

By removing repeated rows of (3) and sorting the remaining rows from $-MNd$ to MNd, we have the linear model rearranged as

$$\tilde{\boldsymbol{z}} = \tilde{\boldsymbol{\Phi}}\boldsymbol{s} + \sigma^2\tilde{\boldsymbol{w}}. \tag{4}$$

It is easy to verify that $\tilde{\boldsymbol{w}} \in \mathbb{R}^{(2MN+1)\times 1}$ is a vector whose elements all equal zero, except the $(MN+1)$th element equals one. The matrix $\tilde{\boldsymbol{\Phi}} \in \mathbb{R}^{(2MN+1)\times K}$ is the steering matrix for a uniform linear array (ULA) with $2MN+1$ sensors and $\tilde{\Phi}_{m,n} = e^{\boldsymbol{j}md\frac{2\pi}{\lambda}\sin(\theta_n)}$ with $m = -MN, -MN+1,\ldots,MN$ and $n = 1,2,\ldots,K$. Therefore, (4) can be regarded as a ULA detecting a coherent source $\boldsymbol{s}$ with deterministic noise term.

III. DOA Estimation Using Continuous Sparse Recovery

DOA estimation with co-prime arrays can be related to the continuous sparse recovery method [6] by a straightforward change of variables. Letting $\tau_k = \frac{d}{\lambda}(1-\sin(\theta_k))$ for all k, the linear model of (4) can be transformed into

$$\begin{aligned} r_n &= e^{-\boldsymbol{j}2\pi n\frac{d}{\lambda}}(\tilde{z}_n - \sigma^2\tilde{w}_n) = e^{-\boldsymbol{j}2\pi n\frac{d}{\lambda}}\sum_{k=1}^{K} s_k e^{\boldsymbol{j}2\pi n\frac{d}{\lambda}\sin(\theta_k)} \\ &= \sum_{k=1}^{K} s_k e^{-\boldsymbol{j}2\pi n\tau_k} = \int_0^1 e^{-\boldsymbol{j}2\pi n\tau} s(d\tau), \end{aligned} \tag{5}$$

where $n = -MN, -MN+1,\ldots,MN-1, MN$. Using the operator $\boldsymbol{F}$ to denote the low frequency measurement operator, we can write $\boldsymbol{r} = \boldsymbol{F}s$, in which $\boldsymbol{r} = [r(-MN),\ldots,r(MN)]^{\mathrm{T}}$ and $s = s(\tau), 0 \le \tau \le 1$, where the measure $s(\tau)$ is given as:

$$s(\tau) = \sum_{k=1}^{K} s_k \delta_{\tau_k}. \tag{6}$$

We use $\mathcal{T} = \{\tau_k, 1 \le k \le K\}$ to denote the support set.

The following convex optimization formula was proposed in [6] to solve the continuous sparse recovery problem:

$$\min_s \|s\|_{\mathrm{TV}} \quad \text{s.t.} \quad \boldsymbol{F}s = \boldsymbol{r}. \tag{7}$$

Total variation minimization is introduced to encourage the sparsity in continuous signals, and is defined as

$$\|s\|_{\mathrm{TV}} = \sup \sum_{j=1}^{\infty} |s(B_j)|,$$

the supremum being taken over all partitions of the set $[0,1]$ into countable collections of disjoint measurable sets B_j.

A theorem about the resolution and degrees of freedom for co-prime arrays can be directly derived using Theorem 1.2 in [6]. Before introducing the theorem, we first define the minimum distance between any two sources as

$$\Delta(\boldsymbol{\theta}) = \min_{\theta_i,\theta_j,\theta_i\neq\theta_j} |\sin(\theta_i) - \sin(\theta_j)|.$$

Theorem III.1. *Consider a co-prime array consisting of two linear arrays with N and $2M$ sensors respectively. The distances between two consecutive sensors are Md for the first array and Nd for the second array, where M and N are co-prime numbers, and $d \le \frac{\lambda}{2}$. Suppose we have K sources located at $\theta_1,\ldots,\theta_K$. If the minimum distance follows*

$$\Delta(\boldsymbol{\theta}) \ge \frac{2\lambda}{MNd},$$

then by solving the convex optimization (7) *with the signal model* (5)*, one can recover the locations θ_k for $k = 1,\ldots,K$ exactly. The maximum number of sources that can be detected is given by*

$$K_{\max} = \frac{MNd}{\lambda}.$$

When the covariance matrix $\boldsymbol{R}_{xx}$ in (2) is approximated using a finite time sample T, we can formulate the following super resolution optimization problem, which considers the noise, as

$$\min_s \|s\|_{\mathrm{TV}} \quad \text{s.t.} \quad \|\boldsymbol{F}s - \boldsymbol{r}\|_2 \le \epsilon. \tag{8}$$

In order to analyze the robustness of (8), a high resolution kernel is introduced in [7] referred to as the Fejér kernel. In our case it has a cut-off frequency $f_h > MN$ as is given by

$$K_h(t) = \frac{1}{f_h}\sum_{k=-f_h}^{f_h}(f_h + 1 - |k|)e^{\boldsymbol{j}2\pi kt}. \tag{9}$$

Theorem III.2. *Consider a co-prime array with the same structure as in Theorem III.1. Let $s(\tau) = \sum_{k=1}^{K} s_k\delta_{\tau_k}$. T time sample points are collected for each receiver. By taking the transformation in* (5) *and solving the optimization* (8) *with s_{opt} as the optimal function, we have that*

$$\|K_h * (s_{\mathrm{opt}} - s)\|_{L_1} \le C_0\frac{f_h^2}{M^2N^2}\epsilon,$$

with probability at least $1 - \alpha e^{-\gamma(\epsilon)T}$ when $\epsilon \le 16\sqrt{2MN+1}\sigma^2$, where $\gamma(\epsilon)$ is a increasing function of ϵ. Here C_0 and α are positive constant numbers.

Remark:
K_h defined in (9) is a low pass filter with cut-off frequency $f_h > MN$. By convolving it with the reconstructed error

$s_{\text{opt}} - s$ we get the reconstruction error details up to the frequency f_h. The proof is a nontrivial extension of the work in [7], and can be found in the journal version of this paper [8].

IV. DOA ESTIMATION VIA SEMIDEFINITE PROGRAMMING

We now derive an optimization framework to reconstruct s for co-prime arrays. For DOA estimation the noise power σ^2 is normally unknown. Thus, a more realistic optimization is reformulated as

$$\min_{s,\sigma^2\geq 0} \|s\|_{\text{TV}} \quad \text{s.t.} \quad \|\boldsymbol{r} - \boldsymbol{F}s - \sigma^2 \boldsymbol{w}\|_2 \leq \epsilon, \tag{10}$$

in which $w_n = \tilde{w}_n e^{-\boldsymbol{j}2\pi n \frac{d}{\lambda}}$. The dual problem takes the form

$$\begin{aligned} \max_{\boldsymbol{u}} \quad & \text{Re}[\boldsymbol{u}^*\boldsymbol{r}] - \epsilon\|\boldsymbol{u}\|_2 \\ \text{s.t.} \quad & \|\boldsymbol{F}^*\boldsymbol{u}\|_{L_\infty} \leq 1, \text{Re}[\boldsymbol{u}^*\boldsymbol{w}] \leq 0. \end{aligned} \tag{11}$$

Here $\boldsymbol{u}$ is the dual variable. Due to the first constraint in (11), the problem itself is still an infinite dimensional optimization. It was shown in [6] that the first constraint can be recast as a semidefinite matrix constraint. Thus the infinite dimensional dual problem is equivalent to the following semidefinite programming (SDP):

$$\begin{aligned} \max_{\boldsymbol{u},\boldsymbol{Q}} \quad & \text{Re}[\boldsymbol{u}^*\boldsymbol{r}] - \epsilon\|\boldsymbol{u}\|_2 \\ \text{s.t.} \quad & \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{u} \\ \boldsymbol{u}^* & 1 \end{bmatrix} \succeq 0, \quad \text{Re}[\boldsymbol{u}^*\boldsymbol{w}] \leq 0, \\ & \sum_{i=1}^{2MN+1-j} \boldsymbol{Q}_{i,i+j} = \begin{cases} 1 & j = 0, \\ 0 & j = 1, 2, \ldots, 2MN. \end{cases} \end{aligned} \tag{12}$$

Solving (12) yields the optimal solution only for the dual problem. The following lemma is introduced to link the solutions of the primal and dual problems.

Lemma IV.1. *Let s_{opt} and $\boldsymbol{u}_{\text{opt}}$ be the optimal solutions of the primal problem* (10) *and dual problem* (12) *respectively. Then*

$$\boldsymbol{F}^*\boldsymbol{u}_{\text{opt}}(\tau) = \text{sgn}(s_{\text{opt}}(\tau))$$

for all τ such that $s_{\text{opt}}(\tau) \neq 0$.

The support set $\mathcal{T}$ can be estimated by root-finding based on the trigonometric polynomial $1 - |\boldsymbol{F}^*\boldsymbol{u}(\tau)|^2 = 0$. Let $\mathcal{T}_{\text{est}}$ denote the estimation of the support sets, and use $\tau_{\text{est}}[i]$ to denote elements in $\mathcal{T}_{\text{est}}$ with $1 \leq i \leq K_{\text{est}}$. A matrix $\boldsymbol{F}^{\text{est}} \in \mathbb{C}^{(2MN+1)\times K_{\text{set}}}$ can be formulated, with measurement $\boldsymbol{r}$ expressed as

$$\boldsymbol{r} = \boldsymbol{F}^{\text{est}}\boldsymbol{s}_0 + \sigma^2\boldsymbol{w} + \boldsymbol{e}, \tag{13}$$

in which $\boldsymbol{s}_0 \in \mathbb{R}^{K_{\text{est}}}$, $\boldsymbol{F}^{\text{est}}_{m,n} = e^{-\boldsymbol{j}md2\pi\tau_{\text{est}}[n]}$ with $m = -MN, MN+1, \ldots, MN$ and $n = 1, \ldots, K_{\text{est}}$, and $\boldsymbol{e}$ is the error term due to the limited number of time samples.

Due to the the numerical issue in the root finding process, the cardinality of $\mathcal{T}_{\text{est}}$ is normally larger than the cardinality of $\mathcal{T}$, i.e., $K_{\text{est}} \geq K$. It is possible in some cases that $K_{\text{est}} \geq 2MN+1$, which leads to an ill-conditioned linear system (13). Sparsity can then be exploited on this signal $\boldsymbol{s}_0$. A convex optimization in the discrete domain can be formulated as

$$\min_{\boldsymbol{s}_0,\sigma^2\geq 0} \|\boldsymbol{s}_0\|_1 \quad \text{s.t.} \quad \|\boldsymbol{r} - \boldsymbol{F}^{\text{est}}\boldsymbol{s}_0 - \sigma^2\boldsymbol{w}\|_2 \leq \epsilon_d. \tag{14}$$

The ϵ_d in (14) is normally chosen to be larger than ϵ in (10) since the noise level is expected to be higher in (13) due to inevitable error from the root finding process. Assuming that the optimization solution of (14) is $\boldsymbol{s}_{\text{est}} \in \mathbb{R}^{K_{\text{est}}}$, the estimation of $\boldsymbol{s}$ in the continuous domain can be represented as

$$\boldsymbol{s}_{\text{opt}} = \sum_{i=1}^{K_{\text{est}}} s_{\text{est}}[i]\delta_{\tau_{\text{est}}[i]}.$$

V. NUMERICAL RESULTS

We consider a co-prime array with 11 sensors. One set of sensors is located at positions $[0, 3, 6, 9, 12]d$, and the second set of sensors is located at positions $[0, 5, 10, 15, 20, 25]d$, where d is taken as half of the wavelength. Thus $N = 5$ and $M = 3$. The first sensors from both sets are collocated. It is easy to show that the correlation matrix generates a virtual array with lags from $-17d$ to $17d$. We compare the continuous sparse recovery (CSR) techniques with MUISC and also with the discrete sparse recovery method (DSR) considering grid mismatches [5]. The MUSIC method in this simulation follows the spatial smoothing technique in [2]. For the discrete sparse recovery method, we take the grid from -1 to 1, with step size 0.005 for $\sin(\theta)$. The noise levels ϵ in the optimization formulas are chosen by cross validation. We consider 15 narrow band signals located at $\sin(\boldsymbol{\theta}) = [-0.8876, -0.7624, -0.6326, -0.5096, -0.3818, -0.2552, -0.1324, -0.0046, 0.1206, 0.2414, 0.3692, 0.4972, 0.6208, 0.7454, 0.8704]$.

A. Estimation Accuracy

In this first numerical example, we verify that continuous sparse recovery increases the degrees of freedom to $O(MN)$ by implementing the coprime arrays' structure. The ϵ for CSR is taken as 5, and ϵ_d is taken as 10 while DSR uses $\epsilon = 10$. In Fig. 1, we use a dashed line to represent the true directions of arrival. Then we test the estimation accuracy of these three methods via Monte Carlo simulations.

Since traditional MUSIC does not yield the DOA of each source directly, we use the Root MUSIC algorithm when calculating the DOA estimation accuracy. For simplicity, we will still refer them both as MUSIC in this section. The number of sources is assumed to be known for the MUSIC algorithm in this simulation, while sparse methods do not assume this a priori.

Figure 2 shows the DOA estimation error with respect to changing SNR after 50 Monte Carlo simulations. The average CPU times for running CSR, DSR and MUSIC are 6.93s, 9.30s, and 1.46s respectively. The estimation error is calculated based on the sine function of the DOAs. We can see that CSR performs better than DSR uniformly. Both sparse recovery methods achieve better DOA estimation accuracy than MUSIC.

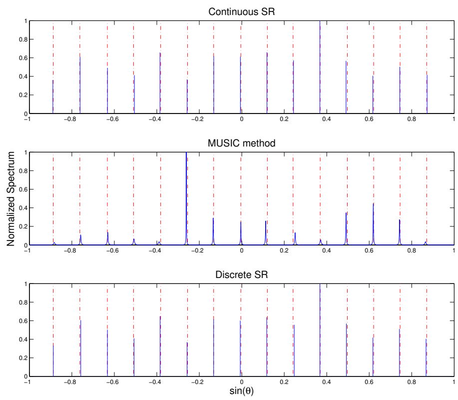

Fig. 1: Normalized spectra for CSR, MUSIC, and DSR, with $T = 500$ and SNR=-10dB.

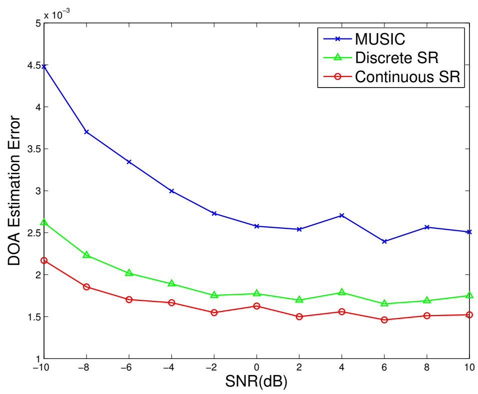

Fig. 2: DOA estimation errors for CSR, MUSIC, and DSR, with $T = 500$.

In Fig. 3 we show that with a changing number of snapshots the proposed CSR also exhibits better estimation accuracy than either DSR or MUSIC. The performance of MUSIC and DSR approach the performance of CSR when the number of snapshots approaches 5000. The average CPU times for running CSR, DSR and MUSIC are 6.50s, 7.91s, and 1.43s respectively.

B. Resolution Ability

Finally we compare the resolution abilities of the MUSIC algorithm and the proposed continuous sparse recovery method. We show that CSR is capable of resolving very closely located signals. In this simulation, two sources are closely located at $-32°$ and $-30°$.The value of ϵ is chosen to be 0.7σ and ϵ_d is set to be 2ϵ in the CSR, where σ is the noise power. MUSIC assumes that the number of sources is known. Even given the number of sources, the MUSIC algorithm fails to resolve the two closely located sources while CSR resolves successfully.

VI. Conclusions

In this work, we extended the mathematical theory of continuous sparse recovery to DOA estimation using co-prime

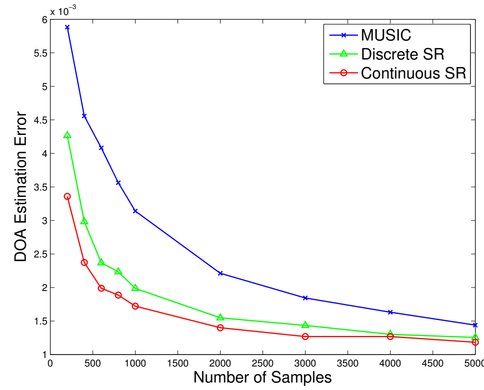

Fig. 3: DOA estimation error for CSR, MUSIC, and DSR, with SNR=-10 dB.

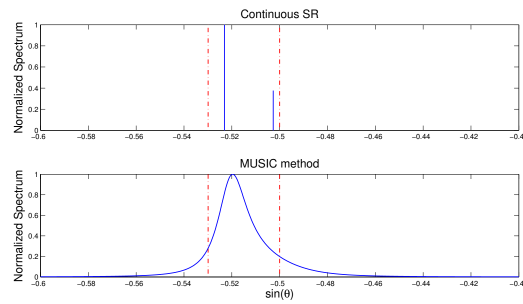

Fig. 4: Resolution capability using CSR and MUSIC algorithm, with SNR=-5 dB, $T = 500$.

arrays. A primal-dual approach was utilized to transform the original infinite dimensional optimization to a solvable semidefinite program. In the future we will extend this idea to random arrays and also generalize the scenario to consider correlations among sources.

References

[1] P. P. Vaidyanathan and P. Pal, "Sparse sensing with co-prime samplers and arrays," *IEEE Trans. Signal Process.*, vol. 59, no. 2, pp. 573–586, 2011.

[2] P. Pal and P. P. Vaidyanathan, "Coprime sampling and the music algorithm," in *Digital Signal Processing Workshop and IEEE Signal Processing Education Workshop (DSP/SPE), 2011 IEEE*, 2011, pp. 289–294.

[3] Y. D. Zhang, M. G. Amin, and B. Himed, "Sparsity-based DOA estimation using co-prime arrays," in *IEEE International Conference on Acoustics, Speech, and Signal Processing, Vancouver, Canada,*, May 2013.

[4] Y. Chi, L. L. Scharf, A. Pezeshki, and A. R. Calderbank, "Sensitivity to basis mismatch in compressed sensing," *IEEE Trans. Signal Process.*, 2011.

[5] Z. Tan and A. Nehorai, "Sparse direction of arrival estimation using co-prime arrays with off-grid targets," *IEEE Signal Process. Lett.*, vol. 21, no. 1, pp. 26–29, 2014.

[6] E. J. Candès and C. F. Granda, "Towards a mathematical theory of super-resolution," *arXiv:1203.5871*.

[7] ——, "Super-resolution from noisy data," *Journal of Fourier Analysis and Applications*, Aug. 2013.

[8] Z. Tan, Y. C. Eldar, and A. Nehorai, "Direction of arrival estimation using co-prime arrays: A super resolution viewpoint," *arXiv:1312.7793*.